

# Math 210C Lecture 11 Notes

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## 1 Additive and Abelian Categories

### 1.1 Additive categories

**Definition 1.1.** An **additive category**  $\mathcal{C}$  is a category such that

1. For  $A, B \in \mathcal{C}$ ,  $\text{Hom}_{\mathcal{C}}(A, B)$  is an abelian group under some operation  $+$  such that for any diagram

$$A \xrightarrow{f} B \begin{array}{c} \xrightarrow{g_1} \\ \xrightarrow{g_2} \end{array} C \xrightarrow{h} D$$

in  $\mathcal{C}$ ,  $h \circ (g_1 + g_2) \circ f = h \circ g_1 \circ f + h \circ g_2 \circ f$ .

2.  $\mathcal{C}$  has a zero object  $0$ .
3.  $\mathcal{C}$  admits finite coproducts.

**Example 1.1.**  $\text{Ab}$  is an additive category.

**Example 1.2.** Let  $R$  be a ring.  $R\text{-Mod}$  is an additive category.

**Example 1.3.** The full subcategory of  $R\text{-Mod}$  of finitely generated  $R$ -modules is additive.

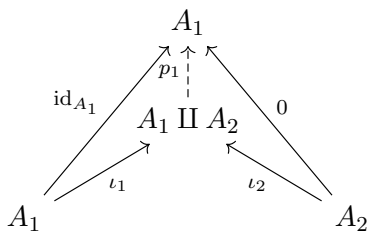
**Example 1.4.** The category of topological abelian groups with continuous homomorphisms is additive.

**Lemma 1.1.** *An additive category  $\mathcal{C}$  admits finite product. In fact, for  $A_1, A_2 \in \mathcal{C}$ , there are natural isomorphisms  $A_1 \times A_2 \cong A_1 \amalg A_2$ . For the inclusion morphisms  $\iota_i : A_i \rightarrow A_1 \amalg A_2$  and projection morphisms  $p_i : A_1 \times A_2 \rightarrow A_i$ , we have  $p_i \circ \iota_i = \text{id}_{A_i}$ ,  $p_i \circ \iota_j = 0$  for  $i \neq j$ , and  $\iota_1 \circ p_1 + \iota_2 \circ p_2 = \text{id}_{A_1 \amalg A_2}$ .*

*Proof.* We have the  $\iota_i$ s. define  $p_i : A_1 \amalg A_2 \rightarrow A_i$  by

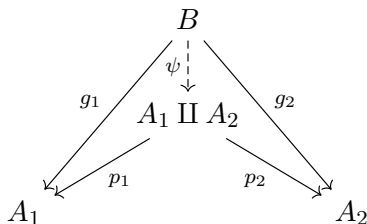
$$p_i \circ \iota_j = \begin{cases} \text{id}_{A_i} & i = j \\ 0 & i \neq j \end{cases},$$

which exists by the universal property of the coproduct:



Then  $(\iota_1 \circ p_1 + \iota_2 \circ p_2) \circ \iota_1 = \iota_1 \implies \iota_1 \circ p_1 + \iota_2 \circ p_2 = \text{id}_{A_1 \amalg A_2}$  by the universal property of the coproduct.

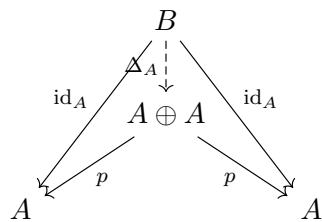
Given  $B \in \mathcal{C}$  and  $g_i : B \rightarrow A_i$ , set  $\psi = \iota_1 \circ g_1 + \iota_2 \circ g_2 : B \rightarrow A_1 \amalg A_2$ . Then  $p_i \circ \psi = g_i$ . If  $\theta : B \rightarrow A_1 \amalg A_2$  is such that  $p_i \circ \theta = g_i$ , then  $\theta = (\iota_1 \circ p_1 + \iota_2 \circ p_2) \circ \theta = \iota_1 \circ g_1 + \iota_2 \circ g_2 = \psi$ .



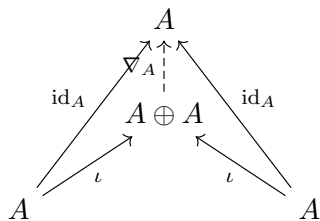
□

**Definition 1.2.**  $A_1 \oplus A_2 = A_1 \amalg A_2 \cong A_1 \times A_2$  is the **biproduct** (or **direct sum**) of  $A_1$  and  $A_2$  in  $\mathcal{C}$ .

**Definition 1.3.** Let  $A$  be an object in an additive category  $\mathcal{C}$ . By the universal property of the product, there is a **diagonal morphism**  $\Delta_A$ :



and a **codiagonal morphism**  $\nabla_A$ :



**Lemma 1.2.** *Let  $\mathcal{C}$  be additive, let  $f, g : A \rightarrow B$  be morphisms, and let  $f \oplus g : A \oplus A \rightarrow B \oplus B$  be induced by the universal property. Then  $f + g = \nabla_B(f \oplus g) \circ \Delta_A$ .*

**Definition 1.4.** A functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  between additive categories is **additive** if for all  $A, B \in \mathcal{C}$ ,  $F : \text{Hom}_{\mathcal{C}}(A, B) \rightarrow \text{Hom}_{\mathcal{D}}(F(A), F(B))$  is a group homomorphism.

**Example 1.5.** If  $\mathcal{C}$  is additive,  $h^A : \mathcal{C} \rightarrow \text{Ab}$  with  $h^A(B) = \text{Hom}_{\mathcal{C}}(B, A)$  is contravariant. Here, the hom sets are now abelian groups. The functor  $h^A$  is additive.

**Example 1.6.** If  $N$  is a right  $R$ -module, let  $t_N : R\text{-mod} \rightarrow \text{Ab}$  send  $t(M) = M \otimes_R N$  and  $f \mapsto f \otimes_R \text{id}_N$ . The functor  $t_N$  is additive.

**Lemma 1.3.**  *$F : \mathcal{C} \rightarrow \mathcal{D}$  is additive if and only if  $F$  preserves biproducts; i.e. for  $A_1, A_2 \in \mathcal{C}$ ,  $F(A_1 \oplus A_2) \rightarrow F(A_1) \oplus F(A_2)$  and  $F(A_1) \oplus F(A_2) \rightarrow F(A_1 \oplus A_2)$  are inverse isomorphisms.*

*Proof.* ( $\implies$ ):  $F(\iota_i) \circ F(p_i) = F(\iota_i \circ p_i) = F(\text{id}_{A_i}) = \text{id}_{F(A_i)}$ . If  $i \neq j$ , then  $F(\iota_i \circ p_j) = F(0) = 0$ . Also,

$$F(\iota_1) \circ F(p_1) + F(\iota_2) \circ F(p_2) = F(\iota_1 \circ p_1 + \iota_2 \circ p_2) = F(\text{id}_{A_1 \oplus A_2}) = \text{id}_{F(A_1 \oplus A_2)}.$$

( $\impliedby$ )  $F$  preserves biproducts, so for  $f, g : A \rightarrow B$  in  $\mathcal{C}$ ,  $F(f \oplus g) = F(f) \oplus F(g)$ . So, by the lemma (twice),

$$\begin{aligned} F(f + g) &= F(\nabla_B \circ (f \oplus g) \circ \Delta_A) \\ &= F(\nabla_B) \circ F(f \oplus g) \circ F(\Delta_A) \\ &= \nabla_{F(B)} \circ (F(f) \oplus F(g)) \circ \Delta_{F(A)} \\ &= F(f) + F(g). \end{aligned} \quad \square$$

**Lemma 1.4.** *A morphism in an additive category that admits kernels (resp. cokernels) is a monomorphism (resp. epimorphism) if and only if it has kernel = 0 (resp. cokernel = 0).*

*Proof.* If  $f : A \rightarrow B$  is a monomorphism, then  $\iota : \ker(f) \rightarrow A$  is a monomorphism such that  $f \circ \iota = 0$ . Since  $f$  is a monomorphism,  $\iota = 0$ . Since  $\iota$  is a monomorphism as well, we get  $\ker f = 0$ .

The other direction is an exercise. □

## 1.2 Abelian categories

**Proposition 1.1.** *Let  $\mathcal{C}$  be an additive category that admits kernels and cokernels. If  $f : A \rightarrow B$  is a morphism in  $\mathcal{C}$ , then*

1.  $\text{im}(f) \cong \ker(B \rightarrow \text{coker}(f))$ ,
2.  $\text{coim}(f) \cong \text{coker}(\ker(f) \rightarrow A)$ .

**Definition 1.5.** An **abelian category** is an additive category that admits all kernels and cokernels and in which every morphism is strict (i.e.  $\text{coim}(f) \rightarrow \text{im}(f)$  is an isomorphism for all  $f$ ).

**Example 1.7.** The category  $R\text{-Mod}$  is abelian. If  $f : A \rightarrow B$  in  $R\text{-mod}$ , then  $\text{coim}(f) = A / \ker(f) \cong \text{im}(f)$ .

**Example 1.8.**  $\text{Grp}$  is not an abelian category because it is not additive. The Hom sets are not groups.

**Example 1.9.** The category of finitely generated  $R$ -modules need not be abelian. If  $I \subseteq R$  is an ideal which is not finitely generated and  $f : R \rightarrow R/I$ , then  $\ker(f) = I$  in  $R\text{-mod}$ , which is not finitely generated.

**Example 1.10.** The category of topological abelian groups is not abelian. Let  $\iota : \mathbb{Q} \rightarrow \mathbb{R}$  be the inclusion map using the discrete topology on  $\mathbb{Q}$  and the usual topology on  $\mathbb{R}$ . Then  $\text{coim}(\iota) = \mathbb{Q}$  and  $\text{im}(\iota) = \mathbb{R}$ , so these are not isomorphic; that is,  $\iota$  is not strict.

**Proposition 1.2.** *If  $\mathcal{C}$  is small and  $\mathcal{D}$  is abelian, then  $\text{Fun}(\mathcal{C}, \mathcal{D})$  is abelian.*