Math 210C Lecture 11 Notes

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1 Additive and Abelian Categories

1.1 Additive categories

Definition 1.1. An **additive category** $\mathcal{C}$ is a category such that

1. For $A, B \in \mathcal{C}$, $\mathrm{Hom}_{\mathcal{C}}(A, B)$ is an abelian group under some operation $+$ such that for any diagram
$$A \xrightarrow{f} B \underset{g_2}{\overset{g_1}{\rightrightarrows}} C \xrightarrow{h} D$$
in $\mathcal{C}$, $h \circ (g_1 + g_2) \circ f = h \circ g_1 \circ g + h \circ g_2 \circ f$.

2. $\mathcal{C}$ has a zero object 0.

3. $\mathcal{C}$ admits finite coproducts.

Example 1.1. Ab is an additive category.

Example 1.2. Let R be a ring. R-Mod is an additive category.

Example 1.3. The full subcategory of R-Mod of finitely generated R-modules is additive.

Example 1.4. The category of topological abelian groups with continuous homomorphisms is additive.

Lemma 1.1. *An additive category $\mathbb{C}$ admite finite product. In fact, for $A_1, A_2 \in \mathcal{C}$, there are natural isomorphisms $A_1 \times A_2 \cong A_1 \amalg A_2$. For the inclusion morphisms $\iota_i : A_i \to A_1 \amalg A_2$ and projection morphisms $p_i : A_i \times A_2 \to A_i$, we have $p_i \circ \iota_i = \mathrm{id}_{A_i}$, $p_i \circ \iota_j = 0$ for $i \neq j$, and $\iota_1 \circ p_1 + \iota_2 \circ p_2 = \mathrm{id}_{A_1 \amalg A_2}$.*

Proof. We have the ι_is. define $p_i : A_1 \amalg A_2 \to A_i$ by
$$p_i \circ \iota_j = \begin{cases} \mathrm{id}_{A_i} & i = j \\ 0 & i \neq j \end{cases},$$

which exists by the universal property of the coproduct:

$$\begin{tikzcd}
& A_1 & \\
& A_1 \amalg A_2 \arrow[u, dashed, "p_1"] & \\
A_1 \arrow[uur, "\mathrm{id}_{A_1}"] \arrow[ur, "\iota_1"'] & & A_2 \arrow[uul, "0"'] \arrow[ul, "\iota_2"]
\end{tikzcd}$$

Then $(\iota_1 \circ p_1 + \iota_2 \circ p_2) \circ \iota_1 = \iota_1 \implies \iota_1 \circ p_1 + \iota_2 \circ p_2 = \mathrm{id}_{A_1 \amalg A_2}$ by the universal property of the coproduct.

Given $B \in \mathcal{C}$ and $g_i : B \to A_i$, set $\psi = \iota_1 \circ g_1 + \iota_2 \circ g_2 : B \to A_1 \amalg A_2$. Then $p_i \circ \psi = g_i$. If $\theta : B \to A_1 \coprod A_2$ is such that $p_i \circ \theta = g_i$, then $\theta = (\iota_1 \circ p_1 + \iota_2 \circ p_2) \circ \theta = \iota_1 \circ g_1 + \iota_2 \circ g_2 = \psi$.

$$\begin{tikzcd}
& B \arrow[d, dashed, "\psi"'] \arrow[ddl, "g_1"'] \arrow[ddr, "g_2"] & \\
& A_1 \amalg A_2 \arrow[dl, "p_1"] \arrow[dr, "p_2"'] & \\
A_1 & & A_2
\end{tikzcd}$$

$\square$

Definition 1.2. $A_1 \oplus A_2 = A_1 \amalg A_2 \cong A_1 \times A_2$ is the **biproduct** (or **direct sum**) of A_1 and A_2 in $\mathcal{C}$.

Definition 1.3. Let A be an object in an additive category $\mathcal{C}$. By the universal property of the product, there is a **diagonal morphism** Δ_A:

$$\begin{tikzcd}
& B \arrow[d, dashed, "\Delta_A"'] \arrow[ddl, "\mathrm{id}_A"'] \arrow[ddr, "\mathrm{id}_A"] & \\
& A \oplus A \arrow[dl, "p"] \arrow[dr, "p"'] & \\
A & & A
\end{tikzcd}$$

and a **codiagonal morphism** ∇_A:

$$\begin{tikzcd}
& A & \\
& A \oplus A \arrow[u, dashed, "\nabla_A"] & \\
A \arrow[uur, "\mathrm{id}_A"] \arrow[ur, "\iota"'] & & A \arrow[uul, "\mathrm{id}_A"'] \arrow[ul, "\iota"]
\end{tikzcd}$$

Lemma 1.2. *Let $\mathcal{C}$ be additive, let $f, g : A \to B$ be morphisms, and let $f \oplus g : A \oplus A \to B \oplus B$ be induced by the universal property. Then $f + g = \nabla_B (f \oplus g) \circ \Delta_A$.*

Definition 1.4. A functor $F : \mathcal{C} \to \mathcal{D}$ between additive categories is **additive** if for all $A, B \in \mathcal{C}$, $F : \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(F(A), F(B))$ is a group homomorphism.

Example 1.5. If $\mathcal{C}$ is additive, $h^A : \mathcal{C} \to \mathrm{Ab}$ with $h^A(B) = \mathrm{Hom}_{\mathcal{C}}(B, A)$ is contravariant. Here, the hom sets are now abelian groups. The functor h^A is additive.

Example 1.6. If N is a right R-module, let $t_N : R - \mathrm{mod} \to \mathrm{Ab}$ send $t(M) = M \otimes_R N$ and $f \mapsto f \otimes_R \mathrm{id}_N$. The functor t_N is additive.

Lemma 1.3. *$F : \mathcal{C} \to \mathcal{D}$ is additive if and only if F preserves biproducts; i.e. for $A_1, A_2 \in \mathcal{C}$, $F(A_1 \oplus A_2) \to F(A_1) \oplus F(A_2)$ and $F(A_1) \oplus F(A_2) \to F(A_1 \oplus A_2)$ are inverse isomorphisms.*

Proof. ($\Longrightarrow$): $F(\iota_i) \circ F(p_i) = F(\iota_i \circ p_i) = F(\mathrm{id}_{A_i}) = \mathrm{id}_{F(A_i)}$. If $i \neq j$, then $F(\iota_i \circ p_j) = F(0) = 0$. Also,

$$F(\iota_1) \circ F(p_1) + F(\iota_2) \circ F(p_2) = F(\iota_1 \circ p_1 + \iota_2 \circ p_2) = F(\mathrm{id}_{A_1 \oplus A_2}) = \mathrm{id}_{F(A_1 \oplus A_2)} .$$

($\Longleftarrow$) F preserves biproducts, so for $f, g : A \to B$ in $\mathcal{C}$, $F(f \oplus g) = F(f) \oplus F(g)$. So, by the lemma (twice),

$$\begin{aligned} F(f+g) &= F(\nabla_B \circ (f \oplus g) \circ \Delta_A) \\ &= F(\nabla_B) \circ F(f \oplus g) \circ F(\Delta_A) \\ &= \nabla_{F_B} \circ (F(f) \oplus F(g)) \circ \Delta_{F(a)} \\ &= F(f) + F(g). \end{aligned}$$

□

Lemma 1.4. *A morphism in an additive category that admits kernels (resp. cokernels) is a monomorphism (resp. epimorphism) if and only if it has kernel $= 0$ (resp. cokernel $= 0$).*

Proof. If $f : A \to B$ is a monomorphism, then $\iota : \ker(f) \to A$ is a monomorphism such that $f \circ \iota = 0$. Since f is a monomorphism, $\iota = 0$. Since ι is a monomorphism as well, we get $\ker f = 0$.

The other direction is an exercise. □

1.2 Abelian categories

Proposition 1.1. *Let $\mathcal{C}$ be an additive category that admits kernels and cokernels. If $f : A \to B$ is a morphism in $\mathcal{C}$, then*

1. $\mathrm{im}(f) \cong \ker(B \to \mathrm{coker}(f))$,
2. $\mathrm{coim}(f) \cong \mathrm{coker}(\ker(f) \to A)$.

Definition 1.5. An **abelian category** is an additive category that admits all kernels and cokernels and in which every morphism is strict (i.e. $\mathrm{coim}(f) \to \mathrm{im}(f)$ is an isomorphism for all f).

Example 1.7. The category R-Mod is abelian. If $f : A \to B$ in R-mod, then $\mathrm{coim}(f) = A/\ker(f) \cong \mathrm{im}(f)$.

Example 1.8. Grp is not an abelian category because it is not additive. The Hom sets are not groups.

Example 1.9. The category of finitely generated R-modules need not be abelian. If $I \subseteq R$ is an ideal which is not finitely generated and $f : R \to R/I$, then $\ker(f) = I$ in R-mod, which is not finitely generated.

Example 1.10. The category of topological abelian groups is not abelian. Let $\iota : \mathbb{Q} \to \mathbb{R}$ be the inclusion map using the discrete topology on $\mathbb{Q}$ and the usual topology on $\mathbb{R}$. Then $\mathrm{coim}(\iota) = \mathbb{Q}$ and $\mathrm{im}(\iota) = \mathbb{R}$, so these are not isomorphic; that is, ι is not strict.

Proposition 1.2. *If $\mathcal{C}$ is small and $\mathcal{D}$ is abelian, then* $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ *is abelian.*